Pergamon

Int. J. Solids Structures Vol. 31, No. 12/13, pp. 1847–1859, 1994
Elsevier Science Ltd
Printed in Great Britain
0020-7683/94 $7.00 + .00

0020-7683(93)E0012-7

DEVELOPMENT OF BEM FOR CONVECTIVE VISCOUS FLOW PROBLEMS

N. TOSAKA and K. KAKUDA
Department of Mathematical Engineering, College of Industrial Technology,
Nihon University, 2-1 Izumi-Cho, 1-chome, Narashino-City, Chiba 275, Japan

Abstract—Three kinds of boundary element approaches for an unsteady flow problem of incompressible viscous fluid are presented. The first approach which is the so-called boundary-domain-type is based on the use of fundamental solution for the only linear differential operator and the Newton–Raphson iterative procedure. The second one is based on the time splitting technique of the governing equations in which the fundamental equations are split into the convection equation and the Stokes equations. The third one is based on the well-known fractional step (FS) method which is one of the time splitting techniques. In order to show the applicability and effectiveness of our approaches, numerical results of the driven cavity flow example are demonstrated through a comparison with other numerical results.

1. INTRODUCTION

A development of efficient computational scheme for unsteady-state viscous fluid flows becomes important in a field of numerical fluid dynamics. Numerical simulations of the viscous fluid flows which are governed by the Navier–Stokes equations have been performed by many researchers using the finite difference method (see Roache, 1972; Peyret and Taylor, 1983) or the finite element method (see Thomasset, 1981; Pironneau, 1989). In addition to the two numerical methods, the boundary element method has been successfully applied to potential problems and solid mechanics problems (see Banerjee and Butterfield, 1981; Brebbia *et al.*, 1984). Applications of the boundary element method to fluid mechanics involving nonlinear problems have been increasingly developed (see Banerjee and Morino, 1990).

There are some integral equation formulations to incompressible viscous fluid flow problems. Wu and his co-workers (Wu and Thompson, 1973; Wu and Wahbah, 1976; Wu, 1982) presented the numerical solution procedure based on the integral equation representation using the velocity and vorticity as field variables and called the integro-differential method. An integral equation approach in terms of the vorticity and stream function was developed by Onishi *et al.* (1984). They presented suitable boundary element procedures for the solution of the vorticity transport equation and the Poisson's equation which relates the stream function to the vorticity. There are some advantages in using this approach of the incompressible Navier–Stokes equations for two-dimensional problems. However, this approach is not generally extensible because the treatment of boundary conditions is not only indirect but also unapplicable to three-dimensional problems. Kakuda and Tosaka (1984) proposed the boundary element approach by reformulating the unsteady Navier–Stokes equations in terms of the only velocity components based on the well-known penalty function method (see Hughes *et al.*, 1979). Unfortunately, it is not clear how to determine the penalty parameter. Skerget *et al.* (1984, 1985) solved the steady laminar flow problems using the vorticity–velocity formulation.

On the other hand, the integral equation approaches based on the so-called primitive variable formulation which adopts the velocity vector and the pressure as field variables have also been proposed by many researchers. This one is the effective formulation because the treatment of boundary conditions is not only direct but also easily extensible to three-dimensional problems. Investigations using this approach can be found, among others, in Oseen (1927), Ladyzhenskaya (1963), Bush and Tanner (1983), Tosaka and Onishi (1985, 1986a), Tosaka *et al.* (1985), Piva *et al.* (1986, 1988), Hebeker (1986), and Dargush and Banerjee (1991a, 1991b).

Tosaka presented the new integral equation formulations for incompressible viscous flows (Tosaka, 1986, 1989), laminar natural convection problems (Tosaka, 1986; Tosaka and Onishi, 1986b; Tosaka and Fukushima, 1986, 1988) and non-Newtonian fluid flow problem (Tosaka and Kakuda, 1990). This approach based on the above formulation has been effectively applied to analyses of steady viscous flow problems (Tosaka and Kakuda, 1986a) and unsteady viscous flow problems (Tosaka and Kakuda, 1986b, 1988a), and was called the boundary-domain-type integral equation method. The integral equations derived from this method were discretized by not only boundary elements but also internal elements. The final system of equations with a full coefficient matrix was solved effectively by using the Newton–Raphson iterative procedure. However, this scheme needs too much computational time and large main memory size in comparison with the finite difference scheme or the finite element scheme.

To overcome this shortcoming, we developed a new approach which was based on the boundary integral equation formulation by making use of the fundamental solution on each subdomain in the whole domain of the problem, and called the generalized boundary element method (Tosaka and Kakuda, 1988b; Kakuda and Tosaka, 1990a). This method has an advantage that some nonlinear effects are taken into consideration in the fundamental solution for the problem. The final system of equations which involves a sparse coefficient matrix was solved implicitly by using a simple iterative procedure.

Moreover, two approaches by means of the boundary elements based on the time splitting technique, which has been successfully developed in the finite element framework (see Benque *et al.*, 1980; Donea *et al.*, 1982), have been proposed by Kakuda and Tosaka (1990b, 1990c). The one is based on the scheme of Benque *et al.* (1980). The fundamental equations were split into the convection equation and the Stokes equations. The non-linear convection equation could be solved implicitly by using a simple iterative procedure. The obtained convection solutions were also used as the initial velocity to solve the Stokes equations using the boundary element method. The other is based on the FS method. In this approach, we adopted the Navier–Stokes equations written in rotational form as the convection term (see Tanahashi *et al.*, 1990). The fundamental equations were split into the convection–diffusion-type equation and linear Euler-type equations. The generalized boundary element method was applied to solve the convection–diffusion-type equation and the Poisson's equation which relates a scalar potential to an auxiliary velocity vector.

In the present chapter, the above mentioned three approaches to solve the unsteady Navier–Stokes equations governing an incompressible viscous fluid flow are presented compactly. These approaches are the boundary-domain-type integral equation formulation, the boundary element one based on Benque's scheme and the generalized boundary element one based on FS scheme. In Section 2 the problem statement is given. The integral equation formulations for three approaches are presented in Section 3. Numerical examples for the driven cavity flow and conclusions are presented in Sections 4 and 5, respectively.

Throughout this chapter, the summation convection on repeated indices is employed. A comma following a variable is used to denote partial differentiation with respect to a space variable, and a dot over a variable denotes differentiation with respect to time.

2. STATEMENT OF PROBLEMS

Let Ω be a bounded domain in Euclidean space with a piecewise smooth boundary Γ. The unit outward normal vector to Γ is denoted by $\mathbf{n}_i$. Also, $\mathscr{T} = (0, T)$ denotes a closed time interval.

The unsteady flow of an incompressible viscous fluid is governed by the following Navier–Stokes equations and continuity equation in dimensionless form:

Navier–Stokes equations,

$$\dot{u}_i + u_j u_{i,j} = -p_{,i} + \frac{1}{Re}(u_{i,jj} + u_{j,ij}) \quad \text{in } \mathscr{T} \times \Omega, \tag{1}$$

continuity equation,

$$u_{i,i} = 0 \quad \text{in } \mathscr{T} \times \Omega, \tag{2}$$

where u_i is the velocity vector, p is the pressure, and Re is the Reynolds number.

In addition to the above set of equations, the following initial and boundary conditions are prescribed:

initial condition,

$$u_i(x, 0) = u_i^0 \quad \text{in } \Omega, \tag{3}$$

boundary conditions,

$$u_i(x, t) = u_i \quad \text{on } \mathscr{T} \times \Gamma_u, \tag{4}$$

$$\tau_i(x, t) = \tau_i \quad \text{on } \mathscr{T} \times \Gamma_\tau, \tag{5}$$

where u_i^0 denotes the given initial velocity, u_i is the velocity vector prescribed on the velocity boundary Γ_u, and $\hat{\tau}_i$ is the traction vector prescribed on the traction boundary Γ_τ.

3. INTEGRAL EQUATION FORMULATIONS

In this section, we shall consider three kinds of integral equation approaches corresponding to the initial-boundary value problems. The first approach is based on the boundary-domain-type integral equation method which is a systematic and useful one presented in Tosaka and Onishi (1985, 1986a). The second one is the derivation of the boundary element equation based on the time splitting technique which is proposed by Benque *et al.* (1980). The third one is the boundary element formulation based on the FS method.

3.1. *Boundary-domain integral equation formulation* (*approach* 1)

3.1.1. *Problem formulation.* In this approach, the differential equations (1) and (2) can be written in matrix form as follows:

$$L_{IJ} U_J = B_I \tag{6}$$

where $[L_{IJ}]$ is the matrix of the linear differential operators appearing in eqns (1) and (2), $\{U_J\}$ is the unknown vector and $\{B_I\}$ denotes the forcing vector given by the nonlinear convection term. The explicit form of eqn (6) is given by Tosaka and Onishi (1986a).

3.1.2. *Integral equation.* In order to derive the integral representation for eqn (6), we start with the following integral identity over the spatial and temporal domain for the weighting function V_{IK}^*:

$$\int_{\mathscr{T}} \int_{\Omega} (L_{IJ} U_J - B_I) V_{IK}^* \, d\Omega \, dt = 0. \tag{7}$$

Integrating by parts over the domain and the time interval, and after some manipulations, we obtain the following set of integral equations:

$$C_{LK}(y) U_K(y, s) = \int_{\mathscr{T}} \int_{\Gamma} u_i(x, t) \Sigma_{iL}^*(x, t; y, s) \, d\Gamma(x) \, dt - \int_{\mathscr{T}} \int_{\Gamma} \tau_i(x, t) V_{iL}^*(x, t; y, s) \, d\Gamma(x) \, dt$$

$$- \int_{\Omega} Re\, u_i(x, 0) V_{iL}^*(x, 0; y, s) \, d\Omega(x) + \int_{\mathscr{T}} \int_{\Omega} B_I(x, t) V_{IL}^*(x, t; y, s) \, d\Omega(x) \, dt, \tag{8}$$

where $C_{LK}(y)$ denotes the shape coefficient tensor which depends generally on both the location of a field point x and the local geometry at the source point y. And also, V^*_{lL} and Σ^*_{lL} are the time-dependent fundamental solution tensor which is presented explicitly in Tosaka and Onishi (1986a) by using Hörmander's method (Hörmander, 1964).

3.1.3. *Discretization and solution procedure*. We consider a discretization by the space-time elements. Let us assume that the boundary Γ is subdivided into n elements and the domain Ω is discretized into m cells. The time interval $\mathscr{T}$ is also subdivided into breaks $t_{k+1} = t_k + \Delta t$ ($k = 0, 1, 2, \ldots$; Δt is the time increment).

The unknowns $u_i(x, t)$ and $\tau_i(x, t)$ over each element and each time step can be approximated by using the set of interpolation functions as follows:

On the boundary element,

$$\left.\begin{aligned} u_i(x,t) &\doteq \psi_Q(t)\phi_N(x)u_i(x_N, t_Q) \\ \tau_i(x,t) &\doteq \psi_Q(t)\phi_N(x)\tau_i(x_N, t_Q) \end{aligned}\right\}. \tag{9}$$

In the interior cell,

$$u_i(x,t) \doteq \psi_Q(t)\varphi_M(x)u_i(x_M, t_Q), \tag{10}$$

in which $\phi_N(x)$ and $\varphi_M(x)$ denote the interpolation functions defined on boundary element and interior cell, respectively, and $\psi_Q(t)$ is the one defined on each time step. The indices N, M and Q refer to the number of nodes within each element, the one of nodes within each cell and the one of time element, respectively.

Substituting of eqns (9) and (10) into (8), we obtain the following discrete form:

$$\Lambda_{ljRQ}u_{jRQ} + G_{liNQ}\tau_{iNQ} = H_{liNQ}u_{iNQ} - C_{liMQ}u_{iMQ} + N_{liMQjKL}u_{iMQ}u_{jKL}, \tag{11}$$

where the shape coefficient matrix Λ_{ljRQ} is taken to be

$$\Lambda_{ljRQ} = \begin{cases} C_{lj}\phi_R(x)\psi_Q(t) & \text{for a point on the boundary,} \\ \delta_{lj}\varphi_R(x)\psi_Q(t) & \text{for a point inside,} \end{cases} \tag{12}$$

in which δ_{lj} denotes the Kronecker's delta, and the other coefficients are given by:

$$\left.\begin{aligned} G_{liNQ} &= \sum_{p=1}^{n} \int_{{}_p\Gamma} \phi_N(x) \int_{t_k}^{t_{k+1}} V^*_{il}(x,t;y,s)\psi_Q(t)\,dt\,d\Gamma(x), \\ H_{liNQ} &= \sum_{p=1}^{n} \int_{{}_p\Gamma} \phi_N(x) \int_{t_k}^{t_{k+1}} \Sigma^*_{il}(x,t;y,s)\psi_Q(t)\,dt\,d\Gamma(x), \\ C_{liMQ} &= \sum_{l=1}^{m} \int_{{}_l\Omega} Re\varphi_M(x)V^*_{il}(x,0;y,s)\psi_Q(0)\,d\Omega(x), \\ N_{liMQjKL} &= \sum_{l=1}^{m} \int_{{}_l\Omega} Re\varphi_M(x)\varphi_{K,j}(x) \int_{t_k}^{t_{k+1}} V^*_{il}(x,t;y,s)\psi_Q(t)\psi_L(t)\,dt\,d\Omega(x). \end{aligned}\right\} \tag{13}$$

Here, we adopt a constant element (i.e., $\psi_Q(t) = 1$) on a time variable within each time step for the unknowns.

Applying eqn (11) to all boundary nodes and interior ones, we finally obtain the following matrix form:

$$\mathbf{H}\mathbf{u}^{k+1} = \mathbf{G}\boldsymbol{\tau}^{k+1} + \mathbf{C}\mathbf{u}^{k} - \mathbf{N}(\mathbf{u}^{k+1})\mathbf{u}^{k+1}, \tag{14}$$

where $\mathbf{u}^{k+1}$ and $\boldsymbol{\tau}^{k+1}$ are the nodal velocity vector and the nodal traction vector at the $(k+1)$-th time step, respectively, $\mathbf{H}$ and $\mathbf{G}$ are the so-called influence matrices, and $\mathbf{N}(\mathbf{u}^{k+1})$ denotes a nonlinear mapping defined with the convection terms.

After applying boundary conditions and initial condition to eqn (14), the equation can be rewritten in the following final form:

$$\mathbf{A}(\mathbf{u})\mathbf{X} = \mathbf{B} \tag{15}$$

where $\mathbf{A}(\mathbf{u})$ is the system matrix which depends on the unknown vector $\mathbf{u}$, $\mathbf{X}$ is the vector of nodal unknown values on the boundary and interior domain, and $\mathbf{B}$ is the known vector.

Since the final system (15) is nonlinear, we must make use of some iterative procedure in order to solve the equation at each time step. To solve eqn (15) accurately and in a stable manner for high Reynolds numbers, we have found that it is effective to utilise the well-known Newton–Raphson method (Tosaka and Kakuda, 1986a, 1986b). Moreover, it is powerful to employ the time-marching scheme (see Brebbia *et al.*, 1984).

The convergence criterion employed herein is given by

$$\left.\begin{aligned} \|\mathbf{R}(\mathbf{X})\| &\leqslant \varepsilon \\ \mathbf{R}(\mathbf{X}) &\equiv \mathbf{A}(\mathbf{u})\mathbf{X}-\mathbf{B} \end{aligned}\right\}, \tag{16}$$

where $\|\cdot\|$ denotes the Euclidean norm and ε is some small positive number.

3.2. *Boundary element formulation based on Benque's scheme* (*approach* 2)

3.2.1. *Problem formulation.* By applying the time splitting technique (Benque *et al.*, 1980) to eqns (1) and (2), we can split the problem into the following two parts:

(a) convection problem

$$\dot{\bar{u}}_i + \bar{u}_j \bar{u}_{i,j} = 0 \quad \text{in } \mathscr{T} \times \Omega, \tag{17}$$

(b) Stokes problem

$$\left.\begin{aligned} \dot{u}_i &= -p_{,i} + \frac{1}{Re}(u_{i,jj} + u_{j,ij}) \\ u_{i,i} &= 0 \end{aligned}\right\} \quad \text{in } \mathscr{T} \times \Omega, \tag{18}$$

where $\bar{u}_i$ denotes the auxiliary velocity vector.

3.2.2. *Integral equation.* We describe the integral representations for eqns (17) and (18). The convection equation (17) is solved via the method of characteristics as follows:

$$\bar{u}_i(x_j, t_{k+1}) = \bar{u}_i(x_j - \bar{u}_j \Delta t, t_k). \tag{19}$$

On the other hand, we apply the boundary element method to solve the Stokes equations (18). The resulting boundary integral equations can be derived as follows (Tosaka and Onishi, 1986a; Tosaka, 1989):

$$\begin{aligned} C_{LK}(y)U_K(y,s) = {} & \int_{t_k}^{t_{k+1}} \int_\Gamma u_i(x,t)\Sigma^*_{iL}(x,t;y,s)\,\mathrm{d}\Gamma(x)\,\mathrm{d}t \\ & - \int_{t_k}^{t_{k+1}} \int_\Gamma \tau_i(x,t)V^*_{iL}(x,t;y,s)\,\mathrm{d}\Gamma(x)\,\mathrm{d}t - \int_\Omega Re\,u_i(x,t_k)V^*_{iL}(x,t_k;y,s)\,\mathrm{d}\Omega(x). \end{aligned} \tag{20}$$

3.2.3. *Discretization and solution procedure.* Taking into consideration the space–time discretization and substituting the approximate forms of eqns (9) and (10) into eqn (20), we obtain the following matrix form:

$$\mathbf{H}\mathbf{u}^{k+1} = \mathbf{G}\boldsymbol{\tau}^{k+1} + \mathbf{C}\mathbf{u}^{k}. \tag{21}$$

Let us mention briefly the solution procedure for this approach. First, we determine the velocity vector (19) by using both a simple iterative procedure and an interpolation with the cubic spline. The convection solutions obtained can be used as an initial velocity to solve eqn (21) step-by-step.

3.3. *Boundary element formulation based on FS scheme (approach* 3)

3.3.1. *Problem formulation.* Now, we adopt the Navier–Stokes equations written in rotational form as the convection term of eqn (1) (see Tanahashi *et al.*, 1990). By applying the semi-implicit scheme to time derivative of the derived equations, the governing equations are given as follows:

$$\frac{u_i^{k+1} - u_i^k}{\Delta t} - e_{ijl} u_j^k \omega_l^k = -H_{,i}^{k+1} + \frac{1}{Re} u_{i,jj}^k \quad \text{in } \Omega, \tag{22}$$

$$u_{i,i}^{k+1} = 0 \quad \text{in } \Omega, \tag{23}$$

where e_{ijl} is the Eddington's epsilon, ω_l^k $(= e_{ijl} u_{l,j}^k)$ is the vorticity vector at kth time step, and H^{k+1} is the Bernoulli's function defined by:

$$H^{k+1} = p^{k+1} + \frac{u_i^k u_i^k}{2} \tag{24}$$

Making use of the auxiliary vector $\bar{u}_i$, and applying the fractional step scheme to eqns (22) and (23), we obtain the following two parts of problem:

(a) convection–diffusion-type problem

$$\frac{\bar{u}_i - u_i^k}{\Delta t} - e_{ijl} u_j^k \omega_l^k = \frac{1}{Re} u_{i,jj}^k, \tag{25}$$

(b) linear Euler-type problem

$$\left.\begin{aligned} u_i^{k+1} &= \bar{u}_i - \Delta t H_{,i}^{k+1} \\ u_{i,i}^{k+1} &= 0 \end{aligned}\right\}. \tag{26}$$

Here, taking the rotation of the first equation in (26) and making use of the Helmholtz resolution, we obtain the following equation:

$$u_i^{k+1} = \bar{u}_i + \Phi_{,i}, \tag{27}$$

in which Φ is the scalar potential. Moreover, taking the divergence of eqn (27) and taking into consideration the continuity equation in (26), we obtain the following Poisson equation in terms of Φ:

$$\Phi_{,ii} = -\bar{u}_{i,i}. \tag{28}$$

Substituting eqn (27) into (26), we can derive the relation between H^{k+1} and Φ as follows:

$$H^{k+1} = -\frac{1}{\Delta t}\Phi. \tag{29}$$

3.3.2. *Integral equations.* In this stage, we describe the integral representations for eqns (25) and (28). In order to achieve the integral equation formulation of eqn (25), we start with the integral identity in a subdomain Ω_e of the whole domain Ω as follows:

$$\int_{\Omega_e}\left(\frac{\bar{u}_\alpha - u_\alpha^k}{\Delta t} - e_{\alpha jl}u_j^k\omega_l^k - \frac{1}{Re}u_{\alpha,jj}^k\right)\delta_{\alpha i}\phi^*\,\mathrm{d}\Omega = 0, \tag{30}$$

in which ϕ^* is the arbitrary scalar function. Integrating by parts over the subdomain, we can rewrite eqn (30) as

$$\int_{\Omega_e}\frac{1}{Re}u_i^k\phi_{,jj}^*\,\mathrm{d}\Omega = \int_{\Omega_e}\left(\frac{\bar{u}_i - u_i^k}{\Delta t} - e_{ijl}u_j^k\omega_l^k\right)\phi^*\,\mathrm{d}\Omega - \int_{\Gamma_e}\frac{1}{Re}u_{i,j}^k n_j\phi^*\,\mathrm{d}\Gamma + \int_{\Gamma_e}\frac{1}{Re}u_i^k\phi_{,j}^* n_j\,\mathrm{d}\Gamma. \tag{31}$$

Here, the scalar function ϕ^* can be chosen as a fundamental solution which satisfies the following differential equation:

$$\phi_{,jj}^* = -\delta(x-y), \tag{32}$$

where $\delta(x-y)$ is the Dirac delta function with the pole at $x = y$. The fundamental solutions are given as follows:

$$\phi^*(x,y) = -\frac{1}{2\pi}\ln r \quad \text{(for two-dimensional case)}, \tag{33}$$

and

$$\phi^*(x,y) = \frac{1}{4\pi r} \quad \text{(for three-dimensional case)}. \tag{34}$$

Substituting eqn (32) into (31), we derive the following integral representation:

$$c(y)\frac{1}{Re}u_i^k(y) = -\int_{\Omega_e}\left(\frac{\bar{u}_i - u_i^k}{\Delta t} - e_{ijl}u_j^k\omega_l^k\right)\phi^*\,\mathrm{d}\Omega + \int_{\Gamma_e}\frac{1}{Re}u_{i,j}^k n_j\phi^*\,\mathrm{d}\Gamma - \int_{\Gamma_e}\frac{1}{Re}u_i^k\phi_{,j}^* n_j\,\mathrm{d}\Gamma. \tag{35}$$

If we suppose that the vorticity vector is piecewise constant in a subdomain, then the weighted residual integral is expressed as

$$\omega_i^k = \frac{1}{\theta}\int_{\Gamma_e} e_{ijl}n_j u_l^k\,\mathrm{d}\Gamma, \tag{36}$$

in which θ takes the area of Ω_e and the volume of Ω_e for the two-dimensional case and three-dimensional case, respectively.

On the other hand, we apply the boundary element method in a subdomain Ω_e to solve the Poisson equation (28). The solution Φ has the following integral representation with the fundamental solution (33) or (34):

$$c(y)\Phi(y) = -\int_{\Omega_e} \bar{u}_i \phi^*_{,i}\,d\Omega + \int_{\Gamma_e} \bar{u}_i n_i \phi^*\,d\Gamma + \int_{\Gamma_e} \Phi_{,j} n_j \phi^*\,d\Gamma - \int_{\Gamma_e} \Phi \phi^*_{,j} n_j\,d\Gamma. \qquad (37)$$

3.3.3. *Discretization and solution procedure.* By applying the boundary element discretization in subdomain to eqn (35), we obtain the following local matrix form:

$$_e\mathbf{M}\frac{_e\bar{\mathbf{u}} - {}_e\mathbf{u}^k}{\Delta t} = {}_e\mathbf{G}_e\mathbf{u}^k_{,n} - {}_e\mathbf{H}_e\mathbf{u}^k + {}_e\mathbf{A}^k. \qquad (38)$$

Moreover, eqn (38) is also reduced as follows:

$$_e\tilde{\mathbf{M}}\frac{_e\bar{\mathbf{u}} - {}_e\mathbf{u}^k}{\Delta t} = {}_e\mathbf{u}^k_{,n} - {}_e\tilde{\mathbf{H}}_e\mathbf{u}^k + {}_e\tilde{\mathbf{A}}^k, \qquad (39)$$

where $_e\tilde{\mathbf{M}}(= {}_e\mathbf{G}^{-1}{}_e\mathbf{M})$ is the lumped mass matrix.

Taking into consideration the equilibrium conditions of $_e\mathbf{u}^k_{,n}$ on each subdomain and setting up eqn (39) for all subdomains, we can obtain the final system of equations as follows:

$$\bar{\mathbf{U}} = \mathbf{U}^k + \Delta t\mathbf{D}^{-1}\mathbf{F}^k, \qquad (40)$$

where $\mathbf{D}$ is the diagonal coefficient matrix and $\mathbf{F}^k$ denotes the known vector which consists of the velocity and vorticity at the kth time step.

On the other hand, the boundary element discretization of eqn (37) is given as follows:

$$_e\mathbf{H}_e\mathbf{\Phi} = {}_e\mathbf{G}_e\mathbf{\Phi}_{,n} + {}_e\mathbf{Q}_e\bar{\mathbf{u}}^k. \qquad (41)$$

Setting up eqn (41) for all subdomains, the final system of equations is obtained as follows:

$$\mathbf{S\Phi} = \bar{\mathbf{F}}, \qquad (42)$$

where $\mathbf{S}$ is the sparse coefficient matrix and $\bar{\mathbf{F}}$ denotes the known vector with respect to the auxiliary vector $\bar{\mathbf{U}}$.

The solution procedure of this approach is given as follows:

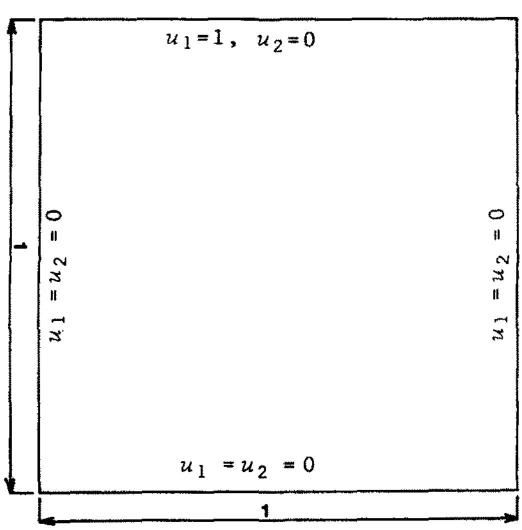


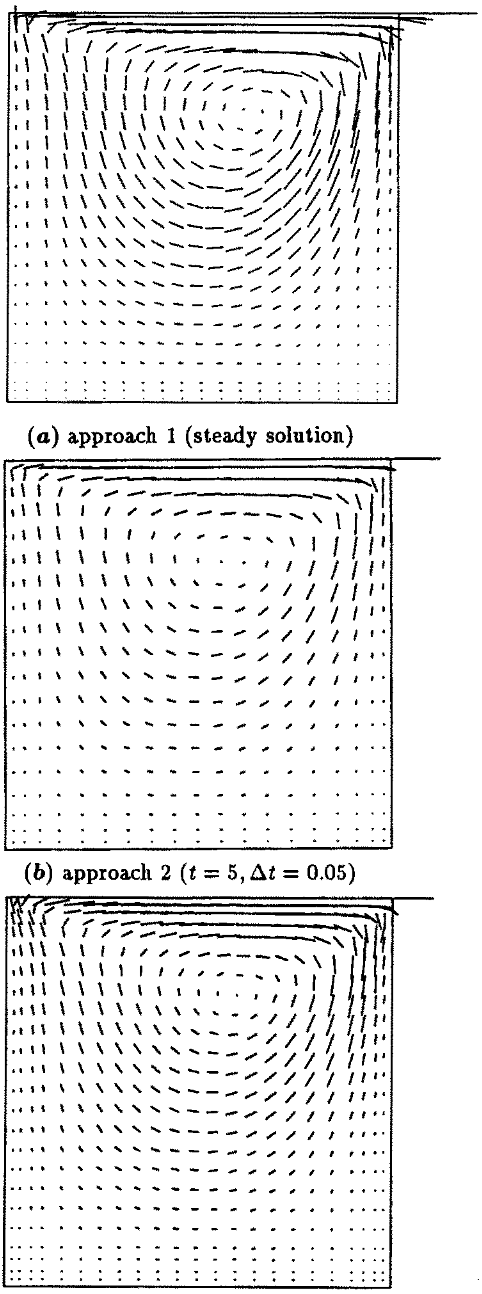
Fig. 1. Driven cavity flow model.

1. Give an initial velocity vector of $\mathbf{U}^k$ and calculate explicitly the auxiliary vector $\bar{\mathbf{U}}$ by eqn (40).
2. Solve the solution $\mathbf{\Phi}$ by applying SOR method to eqn (42).
3. Calculate $\mathbf{U}^{k+1}$ and H^{k+1} at $(k+1)$th time step from the weighted residual statement of eqns (27) and (29), respectively, and go to 1.

4. NUMERICAL EXAMPLES

In order to show the effectiveness and adaptability of three approaches, we demonstrate a recirculation flow in a square cavity driven by a lid sliding at a uniform velocity. The motion of the fluid reaches a steady state gradually.

The geometry and boundary conditions are shown in Fig. 1. The nonuniform element



(*a*) approach 1 (steady solution)

(*b*) approach 2 ($t = 5, \Delta t = 0.05$)

(*c*) approach 3 ($t = 5, \Delta t = 0.005$)

Fig. 2. Velocity vector fields at $Re = 10^2$.

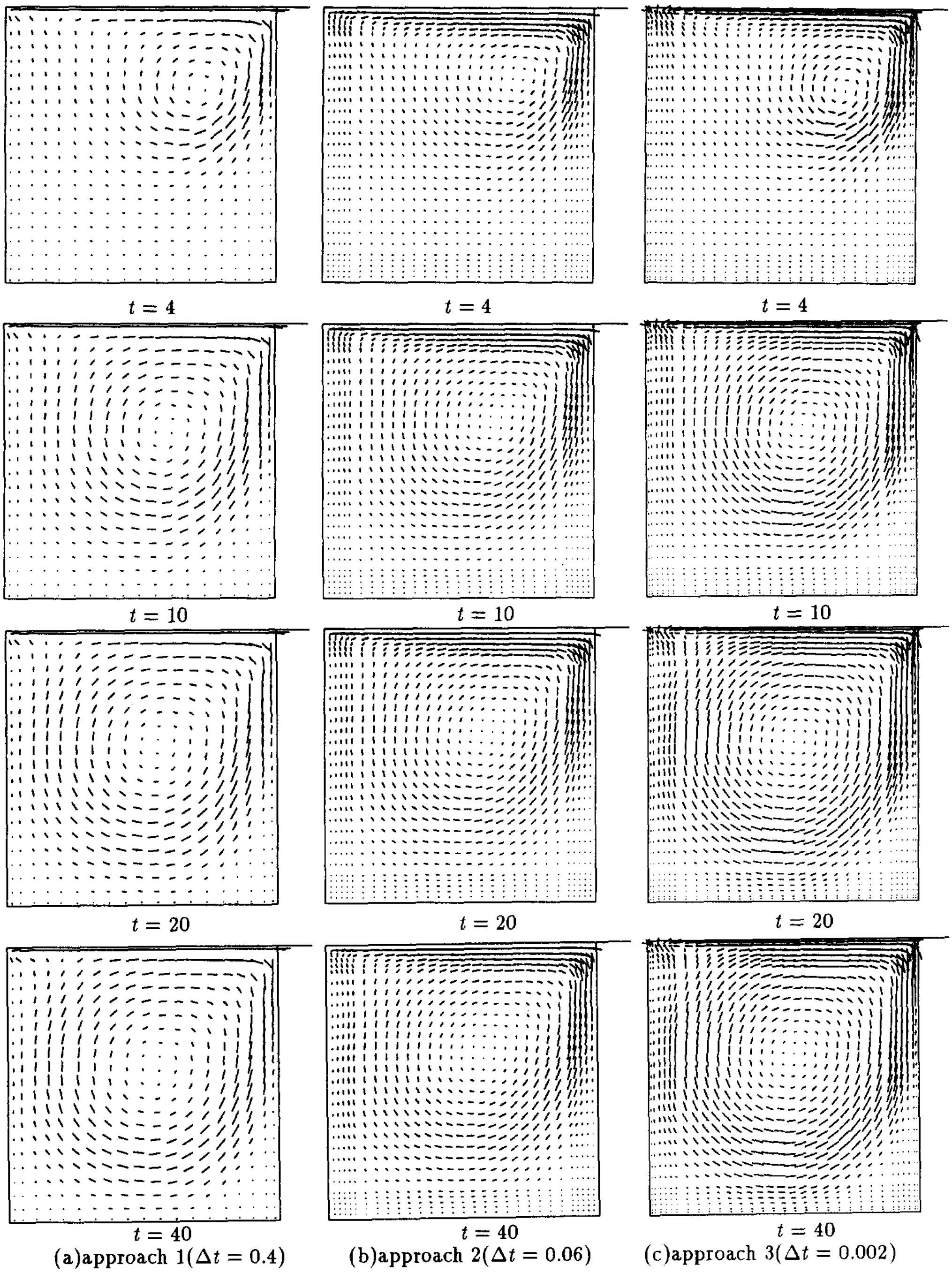


Fig. 3. Velocity vector fields at $Re = 10^3$.

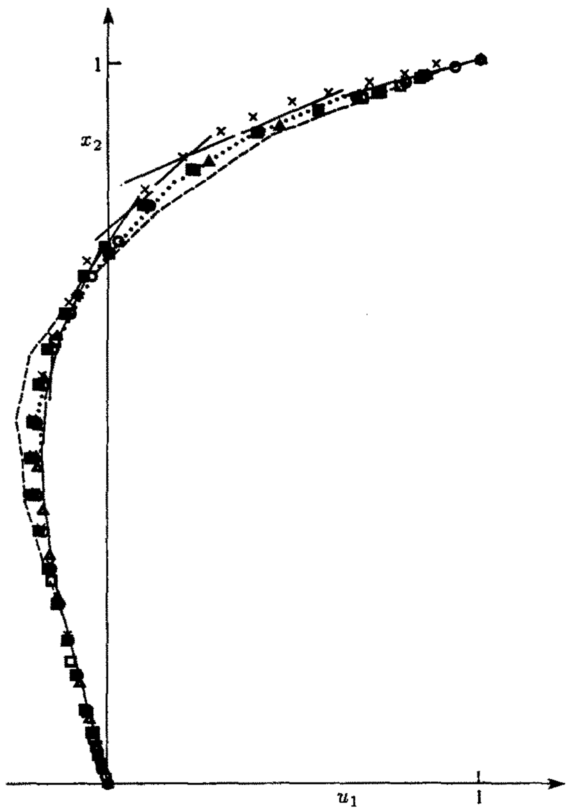


Fig. 4. Comparison of horizontal velocity profiles along vertical centreline ($Re = 10^2$). Present (○ approach 1, 23 by 25 nodes; △ approach 2, 21 × 21; ■ approach 3, 25 × 25); □ Ghia *et al.* (129 by 129 uniform mesh; FDM); × × × Burggraf (40 by 40 uniform: FDM); —— Thomasset (408 elements; FEM); ··· Bercovier and Engelman (Q_2+Q_2 FEM with penalizatio; 12 × 12); --- Borrel ($\omega-\psi P_2+P_2$ FEM; 10 × 10).

which becomes fine near the boundary is utilized. In our numerical performance we adopt the lowest interpolation functions in which the velocity, the traction and the scalar potential are piecewise linear and the pressure, the Bernoulli's function and the vorticity are constant over each element. In the approaches 1 and 2, the constant time element is also adopted. Moreover, the initial velocities are assumed to be zero everywhere in the interior domain.

We show the numerical solutions for $Re = 10^2$ and 10^3. The velocity vector fields for $Re = 10^2$ and 10^3 are shown in Figs 2 and 3, respectively. In Figs 4 and 5, we show the steady horizontal velocity profiles along a vertical centreline inside a cavity, and we compare our results at $t = 5.0$ for $Re = 10^2$ and $t = 40.0$ for $Re = 10^3$ with those obtained by various authors quoted by Thomasset (1981) and Ghia *et al.* (1982). Results for $Re = 10^2$ are in good agreement with those obtained by using different numerical methods. In the case of $Re = 10^3$, the agreement between the results by using approach 2 based on the Benque's scheme and the other ones does not appear satisfactory, but the results using the approaches 1 and 3 are generally comparable to those of Ghia *et al.* (1982).

5. CONCLUSIONS

We have presented three approaches based on the integral equation formulations for the incompressible viscous flow problems. These approaches are the boundary-domain-type integral equation formulation, the boundary element one based on Benque's scheme and the generalized boundary element one based on the FS scheme. The final system of equations derived from the first approach was solved effectively by using the Newton–Raphson iterative procedure at each time step. On the other hand, in both second and third

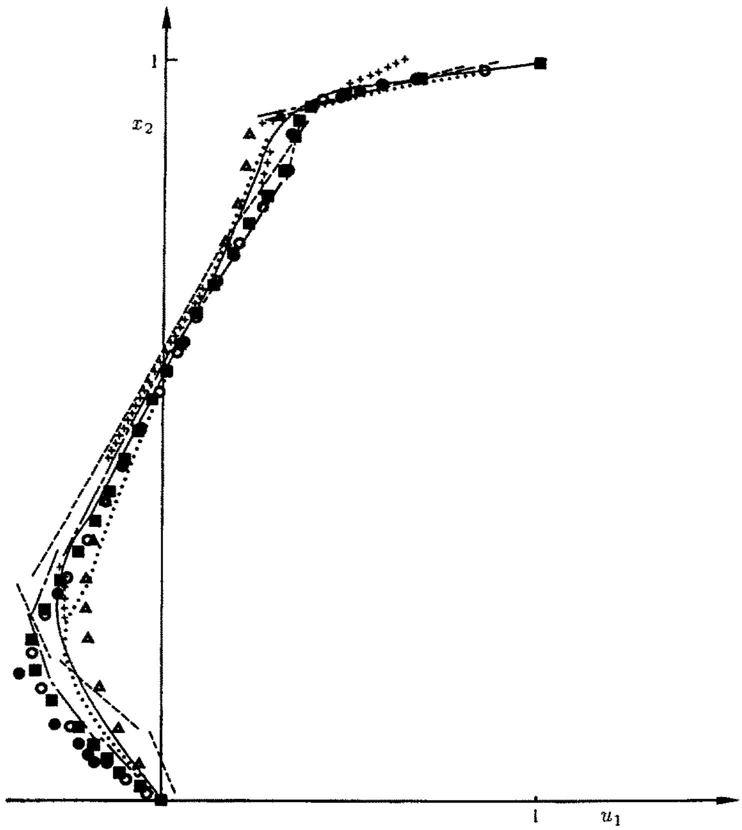


Fig. 5. Comparison of horizontal velocity profiles along vertical centreline ($Re = 10^3$). Present (○ approach 1, 21 by 23 nodes; △ approach 2, 31 × 31; ■ approach 3, 33 × 33); ● Ghia *et al.* (129 by 129 uniform mesh; FDM); —— Nallasamy and Krishaia-Prasad (upwind FDM; 50 × 50); --- Benazeth (mixed $\omega - \psi Q_2 + Q_2$, full upwinding FEM; 10 × 10); -·- Fortin and Thomasset ($Q_2 + Q_2$ elements; 12 × 12); ··· Bercovier and Engelman ($Q_2 + Q_2$ FEM with penalizatio; 12 × 12); + + + Figueroa (mixed "$\psi - \psi_{,ij}$" FEM, with full upwinding; 12 × 12).

approaches the time splitting techniques were introduced and we also applied the simple iterative procedures to solve the final system of equations. Especially, the second approach based on Benque's scheme is unconditionally stable.

Numerical results for the driven cavity flow in two dimensions demonstrated the applicability and effectiveness of the three approaches through a comparison with the other existing results. These approaches can be also extended to three-dimensional problems.

REFERENCES

Banerjee, P. K. and Butterfield, R. (1981). *Boundary Element Methods in Engineering Science*. McGraw-Hill, London.

Banerjee, P. K. and Morino, L. (Eds) (1990). *Boundary Element Methods in Nonlinear Fluid Dynamics, Developments in Boundary Element Methods 6*. Elsevier Applied Science.

Benque, J. P., Ibler, B. and Labadie, G. (1980). A finite element method for Navier–Stokes equations. In *Numerical Methods for Non-Linear Problems* (Edited by C. Taylor, E. Hinton and D. R. J. Owen), pp. 709–720. Pineridge Press, Swansea.

Brebbia, C. A., Telles, T. C. F. and Wrobel, L. C. (1984). *Boundary Element Techniques*. Springer-Verlag.

Bush, M. B. and Tanner, R. I. (1983). Numerical solution of viscous flows using integral equation methods. *Int. J. Numer. Meth. Fluids* **3**, 71–92.

Dargush, G. F. and Banerjee, P. K. (1991a). A boundary element method for steady incompressible thermoviscous flow. *Int. J. Numer. Meth. Engng* **31**, 1605–1626.

Dargush, G. F. and Banerjee, P. K. (1991b). A time-dependent incompressible viscous BEM for moderate Reynolds numbers, *Int. J. Numer. Meth. Engng* **31**, 1627–1648.

Donea, J., Giuliani, S., Laval, H. and Quartapelle, L. (1982). Finite element solution of the unsteady Navier–Stokes equations by a fractional step method. *Comp. Meths Appl. Mech. Engng* **30**, 53–73.

Ghia, U., Ghia, K. N. and Shin, C. T. (1982). High-*Re* solutions for incompressible flow using the Navier–Stokes equations and a multigrid method. *J. Comput. Phys.* **48**, 387–411.

Hebeker, F. K. (1986). On a new boundary element spectral method. In *Innovative Numerical Methods in Engineering* (Edited by R. P. Shaw *et al.*), pp. 311–316. Springer-Verlag.

Hörmander, L. (1964). *Linear Partial Differential Operators*, (second revised printing). Springer-Verlag.

Hughes, T. J. R., Liu, W. K. and Brooks, A. (1979). Finite element analysis of incompressible viscous flows by the penalty function formulation. *J. Comput. Phys.* **30,** 1–60.

Kakuda, K. and Tosaka, N. (1984). Boundary element analysis of the unsteady viscous flows (in Japanese). In *Proc. 1st Japanese National Symp. on Boundary Element Methods*, pp. 241–246.

Kakuda, K. and Tosaka, N. (1990a). The generalized boundary element approach to Burgers' equation. *Int. J. Numer. Methods Engng* **29,** 245–261.

Kakuda, K. and Tosaka, N. (1990b). Boundary element approach to viscous flow problems based on the time splitting technique. In *Boundary Elements XII* (Edited by M. Tanaka, C. A. Brebbia and T. Honma), Vol. 2, pp. 63–72. Springer-Verlag, Tokyo.

Kakuda, K. and Tosaka, N. (1990c). The generalized boundary element approach to viscous flow problems by using the time splitting technique. In *Symp. of IABEM-90*, Università di Roma "La Sapienza", Roma, Italy.

Ladyzhenskaya, O. A. (1963). *The Mathematical Theory of Viscous Incompressible Flow*. Gordon and Breach, New York.

Onishi, K., Kuroki, T. and Tanaka, M. (1984). An application of boundary element method to incompressible laminar viscous flows. *Engng Anal.* **1,** 122–127.

Oseen, C. W. (1927). *Neuere Methoden und Ergebnisse in der Hydrodynamik, Akad.* Verlagsgesellschaft, Leipzig, DDR.

Peyret, R. and Taylor, T. D. (1983). *Computational Methods for Fluid Flow*. Springer-Verlag, New York.

Pironneau, O. (1989). *Finite Element Methods for Fluids*. Wiley, Chichester, New York.

Piva, R., Graziani, G. and Morino, L. (1986). Green's function method for viscous unsteady free surface flows. In *Computational Mechanics 1986* (Edited by G. Yagawa and S. N. Atluri), Vol. 2, pp. XI/123–XI/130. Springer-Verlag, New York.

Piva, R., Graziani, G. and Morino, L. (1988). Boundary integral equation method for unsteady viscous and inviscid flows. In *Advanced Boundary Element Methods* (Edited by T. A. Cruse), pp. 297–304. Springer-Verlag.

Roache, P. J. (1972). *Computational Fluid Dynamics*. Hermosa Publishers, Albuquerque.

Skerget, P., Alujevic, A. and Brebbia, C. A. (1984). The solution of Navier–Stokes equations in terms of vorticity–velocity variables by boundary elements. In *Boundary Elements VI* (Edited by C. A. Brebbia), pp. 4/41–4/56. Springer–Verlag.

Skerget, P., Alujevic, A. and Brebbia, C. A. (1985). Analysis of laminar flows with separation using BEM. In *Boundary Elements VII* (Edited by C. A. Brebbia and G. Maier), pp. 9/23–9/36. Springer-Verlag.

Tanahashi, T., Okanaga, H. and Saito, T. (1990). GSMAC finite element method for unsteady incompressible Navier–Stokes equations at high Reynolds numbers. *Int. J. Numer. Methods Fluids*, Vol. 11, pp. 479–499.

Thomasset, F. (1981). *Implementation of Finite Element Methods for Navier–Stokes Equations*. Springer-Verlag.

Tosaka, N. (1986). Numerical methods for viscous flow problems using an integral equation. In *River Sedimentation* (Edited by S. Y. Wang *et al.*), pp. 1514–1525. University of Mississippi.

Tosaka, N. (1989). Integral equation formulations with the primitive variables for incompressible viscous fluid flow problems. *Comput. Mech.* **4**(2), 89–103.

Tosaka, N. and Fukushima, N. (1986). Integral equation analysis of laminar natural convection problems. In *Boundary Elements VIII* (Edited by M. Tanaka and C. A. Brebbia), Vol. II, pp. 803–812. Springer-Verlag, Tokyo.

Tosaka, N. and Fukushima, N. (1988). Numerical simulations of laminar natural convection problems by the integral equation method. In *Numerical Methods in Thermal Problems* (Edited by R. W. Lewis *et al.*), Vol. V, Part 1, pp. 500–511. Pineridge Press.

Tosaka, N. and Kakuda, K. (1986a). Numerical solutions of steady incompressible viscous flow problems by the integral equation method. In *Innovative Numerical Methods in Engineering* (Edited by R. P. Shaw *et al.*), pp. 211–222. Springer-Verlag.

Tosaka, N. and Kakuda, K. (1986b). Numerical simulations for incompressible viscous flow problems using the integral equation methods. In *Boundary Elements VIII.* (Edited by M. Tanaka and C. A. Brebbia), Vol. II, pp. 813–822. Springer-Verlag, Tokyo.

Tosaka, N. and Kakuda, K. (1988a). Integral equation analysis for high Reynolds number viscous flow problems. In *Proc. 2nd China–Japan Symp. on Boundary Element Methods* (Edited by Du Qinghua), pp. 39–49. Tsinghua University Press, Beijing, China.

Tosaka, N. and Kakuda, K. (1988b). The generalized boundary element method for nonlinear problems. In *Boundary Elements X* (Edited by C. A. Brebbia), Vol. 1, pp. 3–17. Springer-Verlag, Berlin.

Tosaka, N. and Kakuda, K. (1990). Newtonian and non-Newtonian unsteady flow problems. In *Developments in Boundary Element Methods—6* (Edited by P. K. Banerjee and L. Morino), pp. 151–182. Elsevier Applied Science, London.

Tosaka, N. and Onishi, K. (1985). Boundary integral equation formulation for steady Navier–Stokes equations using the Stokes fundamental solutions. *Engng Anal.* **2,** 128–132.

Tosaka, N. and Onishi, K. (1986a). Boundary integral equation formulations for unsteady incompressible viscous fluid flow by time-differencing. *Engng Anal.* **3,** 101–104.

Tosaka, N. and Onishi, K. (1986b). Integral equation method for thermal fluid flow problems. In *Computational Mechanics 1986* (Edited by G. Yagawa and S. N. Atluri), Vol. 2, pp. XI/103–XI/108. Springer-Verlag, New York.

Tosaka, N., Kakuda, K. and Onishi, K. (1985). Boundary element analysis of steady viscous flows based on p-u-v formulation. In *Boundary Elements VII* (Edited by C. A. Brebbia and G. Maier), Vol. 2, pp. 9/71–9/80. Springer-Verlag.

Wu, J. C. (1982). Problems of general viscous flows. In *Developments in Boundary Element Methods—2* (Edited by R. P. Shaw and P. K. Banerjee), pp. 69–109. Applied Science Publishers, London.

Wu, J. C. and Thompson, J. F. (1973). Numerical solution of time-dependent incompressible Navier–Stokes equations using an integro-differential formulation. *Comput. Fluids* **1,** 197–215.

Wu, J. C. and Wahbah, M. M. (1976). Numerical solution of viscous flow equations using integral representation. In *Lecture Notes in Physics*, Vol. 59, pp. 448–453. Springer-Verlag.